Quantum-classical correspondence for resonances on vector bundles

Benjamin Küster

joint work with Tobias Weich, arXiv:1710.04625

Séminaire Problèmes Spectraux en Physique Mathématique

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Outline

$\mathcal{M}$ compact connected Riemannian manifold, no boundary

configuration space of a point particle

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correspondence principle for X and $\Delta_\mathcal{M}$

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Pushforward $\pi_* : \mathrm{C}^\infty(S^*\mathcal{M}) \to \mathrm{C}^\infty(\mathcal{M})$ (fiber integration)

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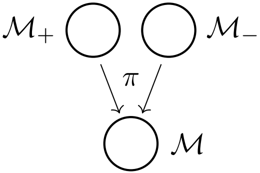

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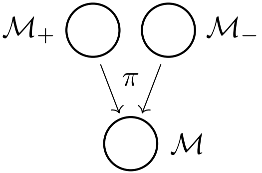

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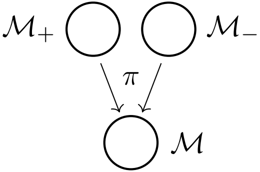

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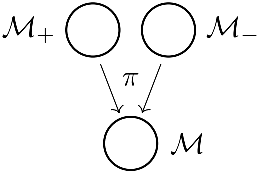

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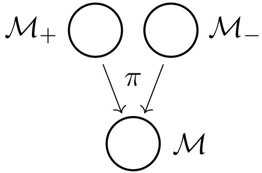

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$$\pi_*(e^{ik\phi}, e^{-ik\phi}) = e^{ik\phi} + e^{-ik\phi} = 2\cos(k\phi) \in \mathrm{Eig}\big(\Delta_{\mathcal{M}}, k^2\big)$$

$$\pi_*(e^{ik\phi}, -e^{-ik\phi}) = e^{ik\phi} - e^{-ik\phi} = 2i\sin(k\phi) \in \mathrm{Eig}\big(\Delta_{\mathcal{M}}, k^2\big)$$

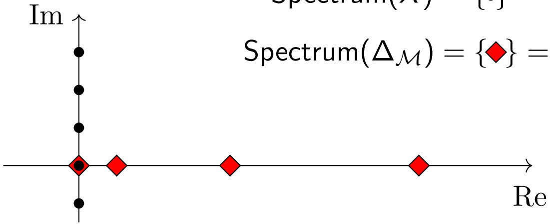
Spectra of $\Delta_{\mathcal{M}}$ and X for $\mathcal{M} = S^1$

Pushfwd $\pi_* : \mathrm{C}^\infty(S^*\mathcal{M}) \to \mathrm{C}^\infty(\mathcal{M})$ induces isomorphisms

$$\mathrm{Eig}(X, ik) \xrightarrow[\cong]{\pi_*} \mathrm{Eig}\big(\Delta_{\mathcal{M}}, k^2\big), \quad k \in \mathbb{Z}$$

$$\text{Spectrum}(X) = \{\bullet\} = i\mathbb{Z}$$

$$\text{Spectrum}(\Delta_{\mathcal{M}}) = \{\blacklozenge\} = \{k^2\}_{k\in\mathbb{N}_0}$$



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An Anosov flow is *chaotic* with positive and negative Lyapunov exponents in E_+ and E_-, respectively

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$\Gamma \subset \mathrm{SO}(n+1,1)_0$ discrete, torsion-free, cocompact

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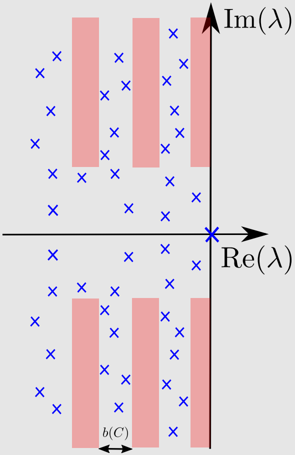
Spectral invariant of *chaotic* dynamical system $(S^*\mathcal{M}, \varphi_t)$

Resonance distribution for pinched curvature

Theorem (Faure-Tsujii 2013)

For pinched sectional curvature

$$-\frac{1}{C} > \kappa > -C \sim -1:$$



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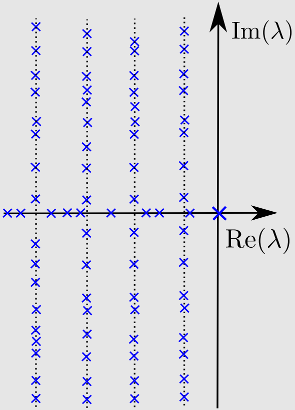

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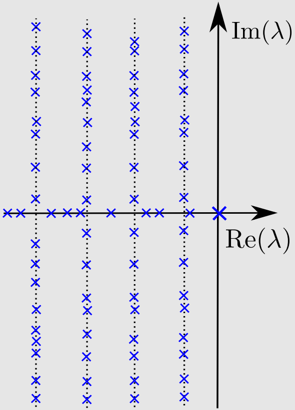

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Proof involves resonant states on vector bundles

Classical resonances on vector bundles

$$\begin{array}{ccc} \mathcal{V} & & \mathcal{W} \\ \downarrow & & \downarrow \\ S^*\mathcal{M} & & \mathcal{M} \end{array} \quad \text{complex v.b. with connections } \nabla^{\mathcal{V}}, \nabla^{\mathcal{W}}$$

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Lemma (Definition)

For $\lambda \in \mathbb{C}$, the set of resonant states on $\mathcal{V}$ is

$$\mathrm{Res}(X_{\mathcal{V}}, \lambda) = \left\{ s \in \mathcal{D}'(S^*\mathcal{M}, \mathcal{V}) : X_{\mathcal{V}}\, s = \lambda\, s,\ \mathrm{WF}(s) \subset E_+^* \right\}.$$

If $\mathrm{Res}(X_{\mathcal{V}}, \lambda) \neq \{0\}$, λ is called classical resonance on $\mathcal{V}$.

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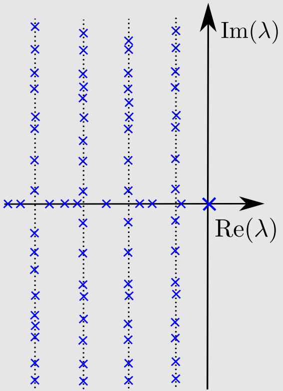
More generally, for $\mathcal{M} = \Gamma\backslash G/K$ Riem. loc. symm. of rk. 1, $\mathcal{V} = \mathcal{V}_\tau$ for unitary rep. τ of certain subgroup $M \subset K$ using $S^*\mathcal{M} = \Gamma\backslash G/M$

Band structure result

For $\mathcal{M} = \Gamma\backslash G/K$ cpt. Riem. loc. symm. of rk. one, $\mathcal{V} = \mathcal{V}_\tau$:

Theorem (T. Weich, B.K., arXiv:1710.04625 (v2 2018))

The classical resonances on $\mathcal{V}$ outside of the real axis lie in exact lines.



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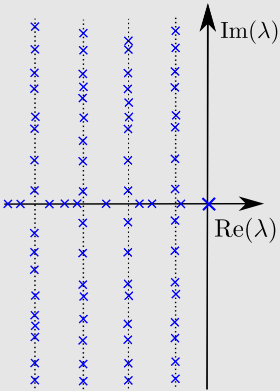
Trivial 1-dim. τ gives scalar result for all compact Riemannian locally symmetric spaces of rank one



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Q Is there an Isom. $\pi_* : \mathrm{Res}^0(X_{\mathcal{V}}, \lambda) \cong \mathrm{Eig}(\Delta_{\mathcal{W}}, \mu(\lambda))$?

Examples of compatible bundles

When $\mathcal{V}$ is a subbundle of $\bigotimes^k T^*(S^*\mathcal{M})$, for example

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Examples of compatible bundles

When $\mathcal{V}$ is a subbundle of $\otimes^k T^*(S^*\mathcal{M})$, for example

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Then there is a natural pushforward

$$\pi_* : \mathcal{D}'(S^*\mathcal{M}, \mathcal{V}) \to \mathcal{D}'(\mathcal{M}, \mathcal{W})$$

Tensor bundles on hyperbolic manifolds

For $\mathcal{M} = \mathcal{H}^{n+1}/\Gamma = \Gamma\backslash\mathrm{SO}(n+1,1)_0/\mathrm{SO}(n+1)$:

Theorem (Dyatlov, Faure, Guillarmou 2013)

For all $\lambda \in \mathbb{C}$ outside a discrete set $\mathcal{A} \subset \mathbb{R}$, there is an iso.

$$\pi_* : \mathrm{Res}^0(X_{\otimes^k_{s,\mathrm{tr}=0} T^*_{S^\perp}(S^*\mathcal{M})}, \lambda) \xrightarrow{\cong} \mathrm{Eig}(\Delta_{\otimes^k_{s,\mathrm{tr}=0} T^*\mathcal{M}}, \mu(\lambda)) \cap \ker \mathrm{div}$$

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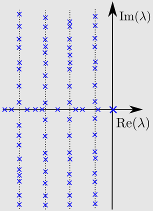
Moreover,

$$\mathrm{Res}^0(X_{\otimes^k_{s,\mathrm{tr}=0} T^*_{S^\perp}(S^*\mathcal{M})}, \lambda) \cong \underbrace{\mathrm{Res}^m(X, \lambda - m)}_{m\text{-th band}}$$



Result for general associated bundles

For $\mathcal{M} = \Gamma\backslash G/K$ cpt. Riem. loc. symm. of rk. one, $\mathcal{V} = \mathcal{V}_\tau$, $\mathcal{W} = \mathcal{W}_\sigma$ for appropriate irreducible rep. τ and σ:

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One has

$$\dim_{\mathbb{C}} \mathrm{Res}^0(X_{\Lambda^p E_+^*}, 0) = \begin{cases} b_p(\mathcal{M}), & p \neq \frac{n}{2}, \\ 2\, b_p(\mathcal{M}), & p = \frac{n}{2}, \end{cases}$$

where $b_p(\mathcal{M}) = \dim_{\mathbb{C}} H^p(\mathcal{M}, \mathbb{C})$ *is the p-th Betti number.*

Similar result proved by Dyatlov and Zworski in dimension 2 and *variable negative curvature*

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Analytic Fredholm theory $\implies$ spectrum(Δ) discrete in $\mathbb{C}$

Anisotropic Sobolev spaces

Faure-Sjöstrand 2011: $\exists\, m \in \mathrm{C}^\infty(T^*(S^*\mathcal{M}), [-1,1])$:

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Resonances as poles of the resolvent

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Poles are classical resonances, residues are eigenspaces with resonant states $\in H^s_{\mathrm{an}} \subset \mathcal{D}'(S^*\mathcal{M})$