The frame flow on hyperbolic 3-manifolds

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joint work with Colin Guillarmou

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Classical problem: decay of matrix coefficients

G real semis. Lie group, real rank 1, finite center, connected

$\Gamma \subset G$ discrete, torsion-free, cocompact

Fix $G = KAN$ (Iwasawa), Haar measure dg

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Gelfand-Fomin '52, Harish-Chandra '58: $\exists\, \delta > 0$ such that $\forall\, u, v \in C^\infty(\Gamma\backslash G)$, as $t \to \infty$:

$$\int_{\Gamma\backslash G} u(\Gamma g e^{tX}) v(\Gamma g)\, dg = \int_{\Gamma\backslash G} u\, dg \int_{\Gamma\backslash G} v\, dg + \mathcal{O}(e^{-\delta|t|})$$

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$\Gamma\backslash G$ is the frame bundle of $\mathcal{Q}$, the right A-action is the frame flow, it is exponentially mixing w.r.t. dg

Goal: re-prove/study this using general microlocal methods

The analytic approach

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For $\lambda \in \mathbb{C}$, $\operatorname{Re}\lambda > 0$, have holomorphic resolvent

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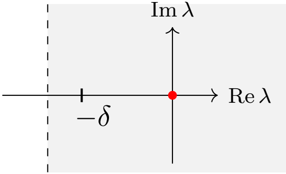
If $R(\lambda) : C^\infty(\Gamma\backslash G) \to \mathcal{D}'(\Gamma\backslash G)$ extends holomorphically to

$$\{\mathrm{Re}\,\lambda > -\delta - \varepsilon\} \setminus \{0\},$$



the frame flow is exponentially mixing w.r.t. dg:

$$\int_{\Gamma\backslash G} u(\Gamma g e^{tX}) v(\Gamma g)\, dg = \int_{\Gamma\backslash G} u\, dg \int_{\Gamma\backslash G} v\, dg + \mathcal{O}(e^{-\delta|t|})$$

Decomposition into irreducible M-modules

$$M := Z_K(A) \subset K, \quad [X, \mathfrak{m}] = 0 \quad (\mathfrak{m} \text{ Lie algebra of } M)$$

$$L^2(\Gamma\backslash G) = \bigoplus_{\substack{(\varrho, V_\varrho) \text{ irred.} \\ M\text{-mod.}}} L^2_\varrho(\Gamma\backslash G) \;\cong\; \bigoplus_\varrho L^2(\Gamma\backslash G/M; \mathcal{V}_\varrho)$$

$$\mathcal{V}_\varrho := \Gamma\backslash G \times_\varrho V_\varrho$$

$$X = \sum_\varrho X_\varrho, \qquad X_\varrho : C^\infty(\Gamma\backslash G/M; \mathcal{V}_\varrho) \circlearrowleft$$

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For $\mathrm{Re}\,\lambda > 0$, have holomorphic resolvents

$$R_\varrho(\lambda) := R(\lambda)|_{L^2_\varrho(\Gamma\backslash G)} = (X_\varrho + \lambda)^{-1} : L^2(\Gamma\backslash G/M; \mathcal{V}_\varrho) \circlearrowleft$$

Pollicott-Ruelle resonances

$X = \sum_\varrho X_\varrho$, geodesic flow on $S\mathcal{Q} = \Gamma \backslash G/M$ is hyperbolic

Liverani '04, Faure-Sjöstrand '11, Dyatlov-Guillarmou '16:

Each $R_\varrho(\lambda) = (X_\varrho + \lambda)^{-1} : C^\infty(S\mathcal{Q}; \mathcal{V}_\varrho) \to \mathcal{D}'(S\mathcal{Q}; \mathcal{V}_\varrho)$

extends meromorphically from $\{\mathrm{Re}\,\lambda > 0\}$ to $\mathbb{C}$

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Quantum-classical correspondence
(Dyatlov-Faure-Guillarmou '14, Weich-K. '18)

Relations to the topology of $\mathcal{Q}$ (Guillarmou-Hilgert-Weich '16, Dang-Rivière '17, Dyatlov-Zworski '17, Weich-K. '19)

Important tools: vector-valued Poisson transforms

Main question

For $\mathrm{Re}\,\lambda > 0$

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Obstructions: 1) Accumulating/infinitely repeated poles

2) Convergence of the infinite sum $\sum_{\varrho} R_{\varrho}(\lambda)$

The case $\mathcal{Q} = \Gamma\backslash\mathbb{H}^3$

$G = \mathsf{PSO}(1,3)$, $K = \mathsf{SO}(3)$, $M = \mathsf{SO}(2) = S^1$

$$\mathbb{Z} \cong \widehat{M} = \{\varrho_n\}_{n\in\mathbb{Z}}, \qquad \varrho_n(z) := z^n, \quad z \in S^1$$

$\mathcal{L}^n := \mathcal{V}_{\varrho_n} = G \times_{\varrho_n} \mathbb{C}$, $n \in \mathbb{Z}$ line bundle

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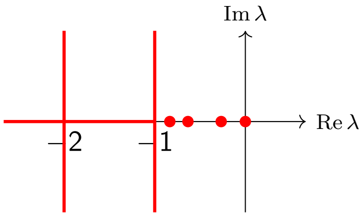
By Dyatlov-Faure-Guillarmou '14:

$\forall\ n \in \mathbb{Z}$, resonances of X_{ϱ_n} on $\mathcal{L}^n$ contained in



Hope for meromorphic extension of $R(\lambda)$ to $\{\mathrm{Re}\,\lambda > -1\}$

Main result

Theorem (Guillarmou-K. 2020)

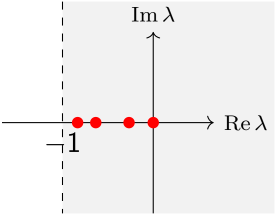
For $G = PSO(1,3)$, the frame flow resolvent

$$R(\lambda) := (X+\lambda)^{-1} : C^\infty(\Gamma\backslash G) \to \mathcal{D}'(\Gamma\backslash G)$$

extends meromorphically to $\{\mathrm{Re}\,\lambda > -1\}$

with only finitely many poles

$$0 = \lambda_0, \lambda_1, \cdots, \lambda_N \in (-1, 0]$$



Methods

To estimate $\|R_{\varrho_n}(\lambda)\|$, treat $n \in \mathbb{Z} \cong \widehat{M}$ as semiclassical parameter alongside of $h \in (0, 1]$

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For a *tensor power function*

$$(0,1] \ni h \mapsto n(h) \in \mathbb{Z}, \qquad |n(h)| = \mathcal{O}(1/h),$$

put

$$\mathcal{L}_h := \mathcal{L}^{n(h)}, \qquad h \in (0,1]$$

Define semiclassical PDOs in $\Psi_h^m(S\mathcal{Q}; \mathcal{L}_h)$ (Charles '92)

Main property of the calculus

For an h-independent line bundle $\mathcal{L}$:

$$P_h, Q_h \in \Psi_h^m(S\mathcal{Q};\mathcal{L}) \implies [P_h, Q_h] \in h\Psi_h^{m-1}(S\mathcal{Q};\mathcal{L}),$$

$$\sigma\Big(\frac{i}{h}[P_h, Q_h]\Big) = \{\sigma(P_h), \sigma(Q_h)\},$$

$\{\cdot,\cdot\}$ Poisson bracket with respect to

$$\omega$$

standard symplectic form on $T^*(S\mathcal{Q})$

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$$\sigma\left(\frac{i}{h}[P_h, Q_h]\right) = \{\sigma(P_h), \sigma(Q_h)\}_h,$$

$\{\cdot,\cdot\}_h$ Poisson bracket with respect to

$$\omega_h := \omega + hn(h)\pi^*\Omega$$

ω standard symplectic form on $T^*(S\mathcal{Q})$
$\pi : T^*(S\mathcal{Q}) \to S\mathcal{Q}$
Ω curvature form of canonical connection on $\mathcal{L}^1$

Outlook

For general $G = KAN$, $M = Z_K(A)$ non-abelian

$\implies \mathcal{V}_\varrho$ in general no line bundle and no tensor power

Need semiclassical calculus for $\mathcal{V}_h$ defined using map

$$(0,1] \ni h \mapsto \chi(h) \in \widehat{M}$$

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Long-term goal: generalization to negatively curved closed Riemannian manifolds