The Ruelle zeta function at zero for nearly hyperbolic 3-manifolds

$\varphi_t : M \to M$ smooth Anosov flow, M cpt. mfld., $TM = E_0 \oplus E_s \oplus E_u$, $E_{s/u}$ orient.

$\zeta_R(\lambda) := \prod_{\gamma \text{ prim.}} (1 - e^{i\lambda T_\gamma})$, $\mathrm{Im}\, \lambda \gg 1$ Ruelle zeta function (untwisted)

γ: primitive closed orbits, T_γ: period of γ

ζ_R extends meromorphically to $\mathbb{C}$ (Giuliani-Liverani-Pollicott 2013 / Dyatlov-Zworski 2016)

Interesting quantity: order of vanishing of ζ_R at $0 \in \mathbb{C}$

$\exists!\ n \in \mathbb{Z}$: $\lambda^{-n} \zeta_R(\lambda)$ holomorphic and non-zero near 0

$n = n(\varphi_t)$ numerical invariant of φ_t determined by period spectrum $\{T_\gamma\}$

Choice of $0 \in \mathbb{C}$ justified by distinguished invariance properties of ζ_R at 0

The invariant $n(\varphi_t)$ can be studied for φ_t in various classes of Anosov flows:

$$\{\text{Anosov flows}\} \supset \{\text{volume-preserving flows}\} \supset \{\text{contact flows}\} \supset \{\text{geodesic flows}\}$$

(M, φ_t)	(M, Ω, φ_t), $\varphi_t^* \Omega = \Omega$,	(M, α), $\iota_X \alpha = 1$, $\iota_X d\alpha = 0$, φ_t : flow of X (Reeb flow of α)	(Σ, g), $M = S_g\Sigma$ unit tangent bundle, φ_t: geodesic flow
$n(\varphi_t)$	$n(\varphi_t) = n(\varphi_t, \Omega)$	$n(\varphi_t) = n(\alpha)$	$n(\varphi_t) = n(g)$

1) Results in dimension 3

Thm. (Dyatlov-Zworski 2017): (M, α) 3-dim. cpt. conn. contact mfld, $\varphi_t : M \to M$ contact Anosov with E_s, E_u orientable

$$\Rightarrow n(\varphi_t) = n(\alpha) = b_1(M) - 2$$

first Betti number of M

In particular, $n(\varphi_t) = n(\alpha)$ independent of α - and of M! - as long as b_1 is fixed

Corollary: $M = S\Sigma$, (Σ, g) cpt. conn. orient. Riem. surface of curvature $K < 0$,

φ_t geodesic flow $\Rightarrow n(\varphi_t) = n(g) = b_1(M) - 2 = -\chi(\Sigma)$ Euler characteristic

In particular, $n(\varphi_t) = n(g)$ independent of g - and of Σ! - for fixed χ

Conversely, $\chi \hat{=}$ genus determined by geodesic length spectrum

Thm. (Cekić - Paternain 2019): (M, α) cpt. conn. 3-dim. contact mfld., $X \in \mathcal{C}^\infty(M; TM)$ Reeb vector field generating an Anosov flow φ_t with E_u, E_s orientable, $\Omega := \alpha \wedge d\alpha$ volume form

$\Rightarrow \exists\, Y \in \mathcal{C}^\infty(M; TM)$, $\varepsilon > 0$: $\mathcal{L}_Y \Omega = 0$,

flow φ_t^τ generated by $X_\tau := X + \tau Y$, $\tau \in \mathbb{R}$

satisfies
$$n(\varphi_t^\tau) = \begin{cases} b_1(M) - 2, & \tau = 0 \\ b_1(M) - 3, & \tau \neq 0,\ |\tau| < \varepsilon \end{cases}$$

$\hookrightarrow n(\varphi_t) = n(\varphi_t, \Omega)$ can <u>jump</u> under small perturbations of φ_t, even for Ω fixed

2) Higher dimensions (mostly dimension 5)

Thm. (Fried 1986): $M = S\Sigma$, (Σ, g_H) hyperbolic $(\mathcal{K} \equiv -1)$ oriented mfld. of dim. $2k+1$, φ_t geodesic flow

$$\Rightarrow n(\varphi_t) = n(g_H) = (2k+2)\beta_0(\Sigma) - 2k\beta_1(\Sigma) + (2k-2)\beta_2(\Sigma) - \ldots + (-1)^k 2\beta_k(\Sigma)$$

$\hookrightarrow$ suggests that $n(\varphi_t) = n(g)$ might be a topological invariant for general g with $\mathcal{K} < 0$, given by the above combination of Betti numbers

Generalizations to variable curvature so far concentrated on Fried's conjecture for the twisted Ruelle zeta function $\zeta_{R,\rho}$ associated with an acyclic (unitary) representation $\rho : \pi_1(M) \to U(\mathbb{C}^r)$, i.e., $b_k(M, E_\rho) = \dim H^k(M; E_\rho) = 0 \;\forall k \geq 0$

de Rham cohomology with values in flat bundle E_ρ def. by ρ

Thm. (Dang-Guillarmou-Rivière-Shen 2018): $M = S\Sigma$, (Σ, g_H) cpt. orient. hyperbolic 3-mfld.,

$\rho : \pi_1(M) \to U(\mathbb{C}^r)$ acyclic unitary rep.,

$X_0 \in \mathcal{C}^\infty(M, TM)$ geodesic vector field

$\Rightarrow \exists\ \mathcal{U} \subset \mathcal{C}^\infty(M, TM)$ open, $X_0 \in \mathcal{U}$, :

$\forall X \in \mathcal{U}$, the flow φ_t generated by X

is Anosov and satisfies

$$\zeta_{R,\rho}(0) = \tau^{\rho}_{Reid}(M)$$

Reidemeister torsion of M,
topological invariant

Our case

ζ_R corresponds to ζ_{R,ρ_0} with $\rho_0 : \pi_1(M) \to \mathbb{C}$

the trivial representation

ρ_0 acyclic $\Leftrightarrow$ $b_k(M) = 0 \quad \forall k \geq 0$

$\hookrightarrow$ The above result has implications on $n(\varphi_t)$ only if all Betti numbers vanish.

Thm. (Cekić - Dyatlov - Paternain - K. 2020): Let (Σ, g) be a compact connected orientable hyperbolic 3-manifold. Then there is an open and dense set $\mathcal{U} \subset \mathcal{C}^\infty(\Sigma, \mathbb{R})$ such that for each $f \in \mathcal{U}$ there is an $\varepsilon > 0$ such that the geodesic flow $\varphi_t^{g_\tau} : S_{g_\tau}\Sigma \to S_{g_\tau}\Sigma$ of the metric $g_\tau := e^{\tau f} g$ satisfies

$$n(g_\tau) = n(\varphi_t^{g_\tau}) = \begin{cases} 4 - 2b_1(\Sigma), & \tau = 0 \\ 4 - b_1(\Sigma), & \tau \neq 0, \ |\tau| < \varepsilon. \end{cases}$$

$\Rightarrow$ If $b_1(\Sigma) > 0$, then the function

$\{\text{metrics on } \Sigma \text{ with } K < 0\} \ni g \longmapsto n(g) \in \mathbb{Z}$ jumps at $g = g_{\text{hyperbolic}}$

along any generic conformal perturbation of $g_{\text{hyperbolic}}$.

$\hookrightarrow$ $n(\varphi_t)$ no longer a topological invariant, seems "sensitive to symmetries"

In particular, $n(\varphi_t)$ can jump under continuous perturbations within the class of contact Anosov flows. More precisely:

<u>Thm.</u> (Cekić - Dyatlov - Paternain - K. 2020): Let $M = S\Sigma$, where Σ is an oriented compact connected hyperbolic 3-manifold. Let $U \subset M$ be a non-empty open set. Then there is an open, dense set $\mathcal{U} \subset C_c^\infty(U, \mathbb{R}) \hookrightarrow C^\infty(M, \mathbb{R})$ such that for every $f \in \mathcal{U}$ $\exists$ $\varepsilon > 0$ such that the Reeb flow φ_t^τ of the contact form $\alpha_\tau := e^{\tau f}\alpha$ satisfies

$$n(\alpha_\tau) = n(\varphi_t^\tau) = \begin{cases} 4 - 2b_1(\Sigma), & \tau = 0, \\ 4 - b_1(\Sigma), & \tau \neq 0, \ |\tau| < \varepsilon. \end{cases}$$

Easier to prove than the previous theorem

$\hookrightarrow$ First results exhibiting jumps of $n(\varphi_t)$ within the classes of contact and geodesic Anosov flows

3) Pollicott-Ruelle resonant states

M cpt., connected, $\varphi_t : M \to M$ Anosov flow with generator $X \in C^\infty(M; TM)$,

$TM = E_0 \oplus E_s \oplus E_u$, $\quad E_s, E_u$ orientable

$T^*M = E_0^* \oplus E_s^* \oplus E_u^*$, $\quad E_0^*(E_s \oplus E_u) = 0$, $E_s^*(E_0 \oplus E_s) = 0$, $E_u^*(E_0 \oplus E_u) = 0$

$\Omega^k := (\Lambda^k T^*M)_{\mathbb{C}}$, $k \in \mathbb{N}_0$, $\Omega^k_0 := \{ \omega \in \Omega^k : \iota_X \omega = 0 \} \cong (\Lambda^k (E_s^* \oplus E_u^*))_{\mathbb{C}}$

$\mathcal{D}'_{E_u^*}(M; \Omega^k) := \{ u \in \mathcal{D}'(M; \Omega^k) : WF(u) \subset E_u^* \}$, similarly with Ω^k_0, E_s^*

$\mathrm{Res}^k_0 := \{ u \in \mathcal{D}'_{E_u^*}(M; \Omega^k_0) : \mathcal{L}_X u = 0 \}$ <u>resonant states</u>, $\quad \dim_{\mathbb{C}} \mathrm{Res}^k_0$: <u>geometric multiplicity</u>

$\mathrm{Res}^{k,\ell}_0 := \{ u \in \mathcal{D}'_{E_u^*}(M; \Omega^k_0) : \mathcal{L}_X^\ell u = 0 \}$ <u>generalized resonant states</u>

$\mathrm{Res}^{k,\infty}_0 := \bigcup_{\ell > 0} \mathrm{Res}^{k,\ell}_0$, $\quad m_{k,0} := \dim_{\mathbb{C}} \mathrm{Res}^{k,\infty}_0$ <u>algebraic multiplicity</u>

By a factorization argument and using microlocal machinery, one finds

$$n(\varphi_+) = \sum_{k=0}^{\dim M - 1} (-1)^k \, m_{k,0}$$ (see Dyatlov-Zworski 2016)

If geometric and algebraic multiplicities coincide, say <u>semisimplicity</u> holds

For $k = \dim M - 1$ and $k=0$, semisimplicity always holds and $m_{0,0} = m_{\dim M-1,0} = 1$

Now, suppose (M,α) contact mfld., $\dim M = 5$, φ_+ contact Anosov.

Then $\wedge\, d\alpha : \mathrm{Res}_0^{1,\infty} \to \mathrm{Res}_0^{3,\infty}$ is an isomorphism

$$\Rightarrow \quad n(\varphi_+) = 2 - 2m_{1,0} + m_{2,0}$$

$\hookrightarrow$ need to understand $m_{k,0}$ for $k=1,2$, requires understanding of semisimplicity

Important tool to study semisimplicity: resonant / co-resonant state pairings

$$\langle\langle\cdot,\cdot\rangle\rangle : C^\infty(M;\Omega_0^k) \times C^\infty(M;\Omega_0^{4-k}) \longrightarrow \mathbb{C}, \quad (u,v) \mapsto \int_M \alpha\wedge u \wedge v$$

extends to

$$\langle\langle\cdot,\cdot\rangle\rangle : \mathcal{D}'_{E_u^*}(M;\Omega_0^k) \times \mathcal{D}'_{E_s^*}(M;\Omega^{4-k}) \longrightarrow \mathbb{C}.$$

$$\mathrm{Res}_{0,*}^k := \{v \in \mathcal{D}'_{E_s^*}(M;\Omega_0^k) : \mathcal{L}_X v = 0\}, \quad \text{similarly: } \mathrm{Res}_{0,*}^{k,\ell}, \quad \mathrm{Res}_{0,*}^{k,\infty}$$

co-resonant states — generalized co-resonant states

<u>Basic observation:</u> semisimplicity holds for k iff $\langle\langle\cdot,\cdot\rangle\rangle$ restricts to a non-degenerate pairing on $\mathrm{Res}_0^k \times \mathrm{Res}_{0,*}^{4-k}$

Involution $\mathcal{J} : M = S\Sigma \to M$, $(x,\xi) \mapsto (x,-\xi)$ induces isomorphism $\mathrm{Res}_0^{k,\ell} \cong \mathrm{Res}_{0,*}^{k,\ell}$ $\forall\, k,\ell$

Important role played by smooth resonant states / smooth representants :

$d\alpha \in \mathrm{Res}_0^2 \cap \mathcal{C}^\infty(M;\Omega_0^2)$

If $u \in \mathrm{Res}_0^k$, $du \in \mathcal{C}^\infty(M;\Omega^{k+1})$, $\exists\ \tilde{u} \in \mathcal{C}^\infty(M;\Omega^k)$, $w \in \mathcal{D}'_{E_u^*}(M;\Omega^{k-1})$:

$$u = \tilde{u} + dw$$

$\hookrightarrow$ get map $\pi_k : \mathrm{Res}_0^k \cap \operatorname{Ker} d \longrightarrow H^k(M;\mathbb{C})$

$$u \longmapsto [\tilde{u}]$$

Hamenstädt 1995 : If $M = S\Sigma$, Σ cpt. orient. 3-dim Riem. mfld. with $\mathcal{K} < 0$, φ_t geodesic flow, and $\dim_{\mathbb{C}} \mathrm{Res}_0^2 \cap \mathcal{C}^\infty(M;\Omega_0^2) > 1$, then $\mathcal{K} = \text{const}$, i.e., Σ is hyperbolic up to constant rescaling

On the other hand, if $M = S\Sigma$ s.t. $\mathcal{K} \equiv -1$, $\exists\ \psi \in \mathrm{Res}_0^2 \cap \mathcal{C}^\infty(M;\Omega_0^2) \setminus (\mathbb{C}\, d\alpha)$

Let now $M = S\Sigma$ with (Σ, g_H) hyperbolic, 3-dim., oriented, connected.

Then we prove:

- Ψ is closed but not exact (Ψ represents the Euler class of $T\Sigma$)
- $\dim \mathrm{Res}^1_0 \cap \ker d = b_1(M)$, $\dim \mathrm{Res}^1_0 = 2\,b_1(M) = m_{1,0}$

  (semisimplicity holds)
- $\dim \mathrm{Res}^2_0 = b_1(M) + 2$, $m_{2,0} = 2\,b_1(M) + 2$

  (semisimplicity <u>fails</u> if $b_1(M) > 0$)
- the range of the map

  $\mathcal{L}_X : \mathrm{Res}^{2,2}_0 \to \mathrm{Res}^2_0$ is equal to $d(\mathrm{Res}^1_0)$
- the existence of Ψ and of non-closed elements in Res^1_0

  is due to the existence of an involution $I : TM \to TM$

  annihilating E_0, rotating by $\frac{\pi}{2}$ in E_s, E_u, commuting with $d\varphi_t$:

  $\Psi|_{(x,v)}(\xi, \eta) = d\alpha|_{(x,v)}(I(x,v)\xi, \eta)$

I corresponds to Hodge star on sphere $S^2 = \partial_\infty \mathbb{H}^3$ at infinity of $\tilde{\Sigma} = \mathbb{H}^3$

Contact perturbations

$\alpha_\tau \in C^\infty(M; T^*M)$, $\tau \in (-\varepsilon, \varepsilon)$, α_0 std. contact form on $M = S\Sigma \cong S^*\Sigma$

$X_\tau \in C^\infty(M; TM)$ Reeb vector field of α_τ; $\varphi_t^\tau : M \to M$ flow of X_τ

For $\varepsilon > 0$ small enough, α_τ is contact and X_τ is Anosov $\forall \tau \in (-\varepsilon, \varepsilon)$

$$\beta := \partial_\tau \alpha_\tau \big|_{\tau=0} \in C^\infty(M; T^*M)$$

__Thm.__: If $\langle\!\langle \mathcal{L}_{X_0}\beta \,\bullet, \bullet \rangle\!\rangle : d(\mathrm{Res}_0^1) \times d(\mathrm{Res}_{0*}^1) \to \mathbb{C}$ is non-degenerate,

then $n(\varphi_t^\tau) = 4 - b_1(\Sigma)$ $\forall \tau \in (-\varepsilon_0, \varepsilon_0) \setminus \{0\}$ for some $\varepsilon_0 > 0$.

Proof methods: Study continuity of resolvents $R_K^\tau(\lambda) := (P_K^\tau - \lambda)^{-1}$, $P_K^\tau := -i\mathcal{L}_{X_\tau}$

on appropriate τ-independent function spaces

$$H_{rG,s}(M;\Omega^k) := e^{-r\,\mathrm{Op}(G)} H^s(M;\Omega^k) \quad , \quad r \geq 0, \quad s \in \mathbb{R},$$

$G(\rho,\xi) = m(\rho,\xi)(1+|\xi|)$ $\quad m$ as in Bonthonneau '20

$$D^{\tau}_{rG,s}(M;\Omega^k) := \{ u \in H_{rG,s}(M;\Omega^k) \mid P^{\tau}_k u \in H_{rG,s}(M;\Omega^k) \}$$

For $\varepsilon_0 > 0$ small $\exists\, C_0$ s.t. $\forall\, r > C_0 + |s|$, $\forall\, \tau \in (-\varepsilon_0, \varepsilon_0)$, $\operatorname{Im} \lambda > -1$,

$P^{\tau}_k - \lambda : D^{\tau}_{rG,s}(M;\Omega^k) \to H_{rG,s}(M,\Omega^k)$ is Fredholm

and the resolvent $R^{\tau}_k(\lambda) : H_{rG,s}(M,\Omega^k) \to H_{rG,s}(M,\Omega^k)$ is bounded locally uniformly in τ, λ outside the set $\{(\tau,\lambda) \in (-\varepsilon_0,\varepsilon_0) \times \mathbb{C} : \lambda \text{ is a resonance of } P^{\tau}_k\}$ which is closed in $(-\varepsilon_0,\varepsilon_0) \times \mathbb{C}$

(Cekić - Paternain '19)

The proof of the main result for contact perturbations then reduces to showing:
$$\forall\, u \in \mathrm{Res}_0^1,\ v \in \mathrm{Res}_{0,*}^1,\ du \neq 0,\ dv \neq 0:$$
$$\mathrm{supp}(\alpha \wedge du \wedge dv) = M.$$

(In unperturbed hyperbolic situation)

Using horocyclic invariance and pullbacks along transport to S^2 at infinity, the statement reduces to the following: $\forall\, g_+, g_- \in \mathcal{D}'(S^2)$, $\mathrm{supp}(g_\pm) = S^2$;
$$\mathrm{supp}\; g_+ \otimes g_- = S^2 \times S^2$$

For the main result on metric perturbations, need to prove:

Given $f \in \mathcal{D}'_{E_s^*}(M) \setminus \{0\}$ s.t. $Xf + 2f = 0$, one has

$$\int_\Sigma a\, \pi_*(ff^*)\, d\mathrm{vol}_\Sigma \neq 0 \qquad \text{for generic } a \in C^\infty(\Sigma), \quad \text{where}$$

$\pi_* : \mathcal{D}'(M) \to \mathcal{D}'(\Sigma)$ is induced by integration over the fiber, $M = S\Sigma$

$f^* := \jmath^* f, \quad \jmath : M \to M, \; (x, \xi) \mapsto (x, -\xi)$

<u>Strategy:</u>

- Assign harmonic 1-forms $u, u^* \in \Omega^1(\Sigma)$ to f, f^* s.t. $u = 0 \Rightarrow f = 0$
- Use a convolution operator to relate u, u^* with $\pi_*(ff^*)$

The operator $Q_s : C_c^\infty(\mathbb{H}^3) \to C^\infty(\mathbb{H}^3)$,

$$f \mapsto \int_{\mathbb{H}^3} K_s(x,z)\, f(x)\, d\mathrm{vol}_{\mathbb{H}^3}(x) =: (Q_s f)(z),$$

$$K_s(x,z) := \left(\cosh d_{\mathbb{H}^3}(x,z)\right)^{-s}$$

extends for $s > 2$ to a smoothing operator $Q_s : \mathcal{D}'(\mathbb{H}^3) \to \mathcal{E}^\infty(\mathbb{H}^3)$

which induces $Q_s : \mathcal{D}'(\Sigma) \to \mathcal{E}^\infty(\Sigma)$.

Main technical result: $\boxed{Q_4 \, \pi_*(ff^*) = \frac{1}{24} \Delta_\Sigma(|U|^2)}$

$\Rightarrow$ If $\pi_*(ff^*) = 0$, then $|U| = \text{const}$ $\underset{\uparrow}{\Rightarrow}$ $U = 0$ $\Rightarrow$ $f = 0$ ⚡

U harmonic 1-form