STUDY GROUP ON PATTERSON-SULLIVAN THEORY

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1. Background on flows

1.1. **General terminology.** Let $\mathcal{M}$ be a metrizable topological space and $\phi^t : \mathcal{M} \to \mathcal{M}$ a continuous flow defined for all $t \in \mathbb{R}$ which has no fixed points.

Definition 1.1.

- The *non-wandering set* $\mathcal{NW}(\phi^t)$ of the flow ϕ^t is the set of all points $x \in \mathcal{M}$ for which there are sequences $x_N \to x$ in $\mathcal{M}$ and $t_N \to +\infty$ in $\mathbb{R}$ such that $\phi^{t_N}(x_N) \to x$.
- The set $\mathcal{P}(\phi^t)$ of *periodic points* of the flow ϕ^t consists of all points $x \in \mathcal{M}$ for which there exists $T > 0$ with $\phi^T(x) = x$.

Note that these sets are ϕ^t-invariant and $\overline{\mathcal{P}(\phi^t)} \subset \mathcal{NW}(\phi^t)$.

Definition 1.2. Let $\mathcal{S} \subset \mathcal{M}$ be a ϕ^t-invariant set and E a continuous vector bundle over $\mathcal{S}$ equipped with a continuous flow $\phi^t_E : E \to E$ lifting ϕ^t over $\mathcal{S}$ and a continuous bundle norm $\|\cdot\|$. Then ϕ^t_E is *uniformly contracting* (resp. *expanding*) on E with respect to $\|\cdot\|$ if there are constants $C, c > 0$ such that for all $p \in \mathcal{S}$ and all $v \in E_p$ one has

$$\left\|\phi^t_E(v)\right\|_{\phi^t(p)} \leq Ce^{-c|t|} \left\|v\right\|_p$$

for all $t \geq 0$ (resp. $t \leq 0$).

Definition 1.3. Suppose that $\mathcal{M}$ is a Riemannian manifold and ϕ^t a C^1-flow with generating vector field $X : \mathcal{M} \to T\mathcal{M}$. Then a ϕ^t-invariant set $\mathcal{S} \subset \mathcal{M}$ is called *hyperbolic* for ϕ^t if $T\mathcal{M}|_{\mathcal{S}}$ admits a Whitney sum decomposition

$$T\mathcal{M}|_{\mathcal{S}} = E^0 \oplus E^s \oplus E^u,$$

where $E_p^0 = \mathbb{R}X(p)$ for all $p \in \mathcal{S}$ and E^s, E^u are $d\phi^t$-invariant continuous subbundles such that $d\phi^t$ is uniformly contracting (resp. expanding) on E^s (resp. E^u) with respect to the Riemannian norm.

1.2. Anosov flows.

Definition 1.4. A C^1-flow ϕ^t on a Riemannian manifold $\mathcal{M}$ is an *Anosov* flow if the entire manifold $\mathcal{M}$ is hyperbolic for ϕ^t.

Remark 1.1. In the literature one often restricts to compact manifolds in the above definition; there seems to be no universal convention.

Theorem 1 (Anosov 1967)**.** *Suppose that ϕ^t is an Anosov flow on a compact Riemannian manifold. Then:*

(1) $\overline{\mathcal{P}(\phi^t)} = \mathcal{NW}(\phi^t)$;

(2) If ϕ^t preserves a measure that is locally absolutely continuous with respect to Lebesgue measure, then ϕ^t is ergodic *with respect to this measure, i.e., every ϕ^t-invariant measurable subset $\mathcal{S} \subset \mathcal{M}$ satisfies either $\mathrm{vol}(\mathcal{S}) = \mathrm{vol}(\mathcal{M})$ or $\mathrm{vol}(\mathcal{S}) = 0$;*

Theorem 2 (Anosov 1967, Moser 1969, Robbin 1971)**.** *Suppose that ϕ^t is an Anosov flow on a compact Riemannian manifold $\mathcal{M}$ with generating vector field X. Then the dynamical system $(\mathcal{M}, \phi^t)$ is* structurally stable, *i.e., for every $\varepsilon > 0$ there is a $\delta > 0$ such that for every vector field X' on $\mathcal{M}$ with $\|X - X'\|_{C^1} < \delta$ there is a homeomorphism $h : \mathcal{M} \to \mathcal{M}$ such that*

(1) $\mathrm{dist}(p, h(p)) < \varepsilon$ for all $p \in \mathcal{M}$;

(2) h intertwines the oriented orbits of ϕ^t with the oriented orbits of the flow $(\phi')^t$ generated by X'.

Example 1.5. (1) The geodesic flow on the unit tangent bundle of a negatively curved closed Riemannian manifold is Anosov.

(2) Let $\mathbb{X} = \Gamma\backslash G/K$ be a locally symmetric space of rank one and fix an Iwasawa decomposition $G = KAN$. The unit tangent bundle of $\mathbb{X}$ can we written as

$$S^1\mathbb{X} = \Gamma\backslash G/M,$$

where $M = Z_K(A) \subset K$ is compact. The geodesic flow ϕ^t on $S^1\mathbb{X}$ is the action of $A \cong \mathbb{R}$ induced by right multiplication. Recall the Bruhat decomposition

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n} \oplus \theta\mathfrak{n},$$

where $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{m} = \mathrm{Lie}(M)$, $\mathfrak{a} = \mathrm{Lie}(A)$, $\mathfrak{n} = \mathrm{Lie}(N)$, and $\theta : \mathfrak{g} \to \mathfrak{g}$ is the Cartan involution defining K. Equipping G with a left-G-invariant and right-M-invariant metric, $S^1\mathbb{X}$ becomes a Riemannian manifold and the Bruhat decomposition induces a splitting

$$T(S^1\mathbb{X}) = (\Gamma\backslash G \times \mathfrak{a}) \oplus (\Gamma\backslash G \times_{\mathrm{Ad}(M)} \mathfrak{n}) \oplus (\Gamma\backslash G \times_{\mathrm{Ad}(M)} \theta\mathfrak{n}).$$

Since for any non-zero element $X \in \mathfrak{a}^+$ and any $Y \in \mathfrak{g}_\alpha$ with $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a})$ we have

$$\mathrm{Ad}(e^{tX})Y = e^{t\alpha(X)}Y$$

and

$$\mathfrak{n} = \bigoplus_{\alpha\in\Sigma^+} \mathfrak{g}_\alpha, \qquad \theta\mathfrak{n} = \bigoplus_{-\alpha\in\Sigma^+} \mathfrak{g}_\alpha,$$

we see that the above splitting makes $S^1\mathbb{X}$ hyperbolic for ϕ^t and thus ϕ^t is an Anosov flow.

1.3. **Axiom A flows.**

Definition 1.6. A smooth flow ϕ^t on a manifold $\mathcal{M}$ is an *Axiom A* flow if it has the following properties:

(1) $\mathcal{NW}(\phi^t)$ is compact;

(2) $\mathcal{NW}(\phi^t)$ is hyperbolic for ϕ^t with respect to some (hence any) continuous norm on $T\mathcal{M}|_{\mathcal{NW}(\phi^t)}$;

(3) $\overline{\mathcal{P}(\phi^t)} = \mathcal{NW}(\phi^t)$.

Example 1.7. Let $\mathbb{X} = \Gamma\backslash G/K$ be a convex-cocompact locally symmetric space of rank one. Then the geodesic flow on $S^1\mathbb{X} = \Gamma\backslash G/M$ (with the notation as in Example 1.5) is an Axiom A flow.

Definition 1.8. A compact ϕ^t-invariant set $\mathcal{K} \subset \mathcal{M}$ is *locally maximal* for the flow ϕ^t if there is a neighborhood $\mathcal{U} \subset \mathcal{M}$ of $\mathcal{K}$ such that

$$\mathcal{K} = \bigcap_{t \in \mathbb{R}} \phi^t(\mathcal{U}).$$

Definition 1.9. A hyperbolic set $\mathcal{K}$ for the flow ϕ^t is *basic* if it is locally maximal for ϕ^t, the flow $\phi^t|_{\mathcal{K}}$ is topologically transitive (i.e., $\mathcal{K}$ contains a dense ϕ^t-orbit), and $\mathcal{K}$ is the closure in $\mathcal{M}$ of the set of periodic points of $\phi^t|_{\mathcal{K}}$.

Theorem 3 ("Spectral decomposition" of the non-wandering set, Smale 1967)**.** *If ϕ^t is an Axiom A flow, then its non-wandering set is a finite disjoint union of basic hyperbolic sets.*

1.4. **Gromov flow spaces.** The following definition generalizes the concept of the geodesic flow on the unit tangent bundle of a compact negatively curved Riemannian manifold, motivated by the fact that the fundamental group of such a manifold is hyperbolic.

Definition 1.10 ([Gro87, Sec. 8.3], [Min05, Thm. 60])**.** A *Gromov-geodesic flow* of Γ is a proper hyperbolic metric space $\widehat{\Gamma}$ endowed with a fixed-point free flow $(\Phi^t)_{t \in \mathbb{R}}$, an isometric involution ι, and an isometric Γ-action with the following properties:

(1) The Γ-action commutes with ι and Φ^t.

(2) The involution ι anti-commutes with Φ^t, i.e., $\iota \circ \Phi^t = \Phi^{-t} \circ \iota$.

(3) The orbit maps $\Gamma \to \widehat{\Gamma}$ are quasi-isometries. In particular, the Γ-action on $\widehat{\Gamma}$ is properly discontinuous and cocompact, and there is a homeomorphism $\partial_\infty \widehat{\Gamma} \cong \partial_\infty \Gamma$. The latter is canonical in the sense that it is independent of the choice of the Γ-orbit.

(4) The orbit maps $\mathbb{R} \to \widehat{\Gamma}$ of the flow Φ^t are quasi-isometric embeddings.

(5) The map

$$\begin{aligned} \tau : \widehat{\Gamma} &\longrightarrow \partial_\infty \Gamma^{(2)} = \partial_\infty \widehat{\Gamma}^{(2)} \\ x &\longmapsto \Big(\underbrace{\lim_{t\to-\infty} \Phi^t(x)}_{=:\tau_+(x)}, \underbrace{\lim_{t\to+\infty} \Phi^{-t}(x)}_{=:\tau_-(x)} \Big) \end{aligned}$$

induces a homeomorphism

$$\widehat{\Gamma}/\mathbb{R} \cong \partial_\infty \Gamma^{(2)}.$$

By [Gro87, Thm. 8.3.C] there exists a Gromov-geodesic flow of Γ. More details see [Min05, pp. 405–406].

Example 1.11. Let $\Gamma = \pi_1(M)$ be the fundamental group of a compact negatively curved Riemannian manifold M. Then the unit tangent bundle $\widetilde{\mathcal{M}} := S^1\widetilde{M} \subset T\widetilde{M}$ of the universal cover $\widetilde{M}$ of M, equipped with the lifted geodesic flow $\Phi^t = \widetilde{\varphi}^t$, the Γ-action given by the derivatives of the Deck transformations, and the involution $\iota(x, v) := (x, -v)$ is a Gromov flow space of Γ.

2. Anosov representations

Let G be a non-compact connected semisimple real Lie group with finite center.

2.1. **Rank one convex-cocompactness.** Suppose that $\mathrm{rk}(G) = 1$, i.e., $\dim A = 1$ in an Iwasawa decomposition $G = KAN$. Geometrically, the maximal flats in the Riemannian symmetric space

$$\widetilde{\mathbb{X}} := G/K$$

are 1-dimensional. Consider a discrete subgroup $\Gamma \subset G$ and put

$$\mathbb{X} := \Gamma\backslash\widetilde{\mathbb{X}}.$$

Γ discrete, K compact $\implies$ Γ acts properly discontinuously on G/K. So $\mathbb{X}$ is a smooth manifold if Γ is torsion-free and an orbifold otherwise. We have the limit set

$$\Delta_\Gamma := \{\text{accumulation points of } \Gamma\text{-orbits}\} \subset \partial_\infty \mathbb{X},$$

the description

$$\partial_\infty \mathbb{X} = G/P, \qquad P = MAN, \quad M = Z_K(A),$$

and the unit tangent bundle (in the orbifold sense if Γ has torsion) with the geodesic flow

$$T^1\mathbb{X} = \Gamma\backslash G/M, \qquad \phi^t(\Gamma g M) := \Gamma g e^{tX} M,$$

where $X \in \mathfrak{a}^+ \subset \mathfrak{a} = \mathrm{Lie}(A)$ with $\|X\| = 1$, and its non-wandering set

$$\mathcal{NW}(\phi^t) \subset \Gamma\backslash G/M.$$

Again: Γ discrete, M compact $\implies$ Γ acts properly disc. on G/M.

Theorem 4 (C.f. [Kas13]). *The following statements are equivalent:*

(1) There exists a non-empty Γ-invariant convex set $S \subset \widetilde{\mathbb{X}}$ on which Γ acts cocompactly;

(2) Γ acts cocompactly on the convex hull $\mathrm{Conv}(\Lambda_\Gamma) \subset \widetilde{\mathbb{X}}$;

(3) The closure of the union of all closed geodesics in $\mathbb{X}$ is compact;

(4) The non-wandering set $\mathcal{NW}(\phi^t)$ of the flow ϕ^t is compact;

(5) ϕ^t is an Axiom A flow;

(6) Γ is finitely generated and the inclusion $\Gamma \to G$ is a quasi-isometric embedding;

(7) Γ is finitely generated and for some word metric d_Γ on Γ there are $c, C > 0$ such that

$$d_{\widetilde{\mathbb{X}}}(\gamma K, K) \geq c\, d_\Gamma(\gamma, e) - C \qquad \forall\, \gamma \in \Gamma,$$

where $d_{\widetilde{X}}$ is the Riemannian distance in $\widetilde{\mathbb{X}}$;

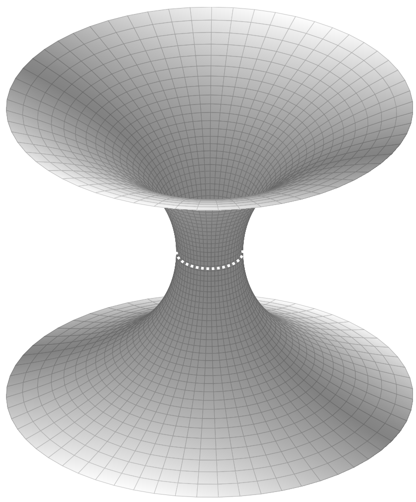
(8) Γ is hyperbolic and there exists a continuous, injective, and Γ-equivariant map

$$\xi : \partial_\infty \Gamma \to G/P = \partial_\infty \mathbb{X}.$$

Definition 2.1. Γ is called *convex-cocompact* if the above conditions hold.

Remark 2.1. Γ being convex-cocompact is *not* equivalent to ϕ^t being an Anosov flow.

Example 2.2.
- Every cocompact Γ is convex-cocompact.
- Every finite Γ is convex-cocompact.
- A free convex-cocompact group is called *Schottky group*. They can be explicitly constructed using a "ping-pong argument".
- Simplest example: $G = \mathrm{SL}(2,\mathbb{R})$, $\Gamma \cong \mathbb{Z}$, $\widetilde{\mathbb{X}} = \mathbb{H}^2$, $\mathbb{X}$ =cylinder



2.2. **Convex-cocompactness in higher rank.**

Idea: Generalize Definition 2.1 to G of higher rank.

Problem: Which of Items (1) – (8) in Thm. 4 generalizes, and how ?

From now on no restriction on the rank of G.

Theorem 5 (Kleiner-Leeb 2006)**.** *Let $G_1 \subset G$ be the product of all simple factors of G of real rank 1 and $G_{\geq 2} \subset G$ the product of all simple factors of ranks ≥ 2. Let $\Gamma \subset G$ be a Zariski-dense discrete subgroup preserving a closed, convex subset $\mathcal{C} \subset \widetilde{\mathbb{X}}$ and acting cocompactly on it. Then Γ is a product of convex-cocompact subgroups of the rank 1 factors in G_1 and a uniform lattice in $G_{\geq 2}$.*

$\Longrightarrow$ Items (1), (2) of Thm. 4 generalize uninterestingly to $\mathrm{rk}(G) \geq 2$.

What about other Items?

As before, let $G = KAN$, $M = Z_K(A)$, $\mathfrak{a} = \mathrm{Lie}(A)$, and $\mathfrak{a}^+ \subset \mathfrak{a}$ a closed Weyl chamber.

Definition 2.3 (Quint 2005)**.** Let $C \subset \mathfrak{a}$ be an open cone.

- A point $x \in \Gamma\backslash G/M$ is *C-conservative* if there is a sequence $X_n \in \mathfrak{a}$ with $X_n \to \infty$ such that $x\exp(X_n) \in \Gamma\backslash G/M$ is bounded and $\frac{X_n}{\|X_n\|}$ converges to a point in C.
- The *C-conservative set* $\Omega_C \subset \Gamma\backslash G/M$ of the Weyl chamber flow on $\Gamma\backslash G/M$ is the closure of the set of *C-conservative* points.

Remark 2.2. If $\mathrm{rk}(G) = 1$, there are only the two open cones $\pm\mathring{\mathfrak{a}}^+$ and

$$\Omega_{\mathring{\mathfrak{a}}^+} \cap \Omega_{-\mathring{\mathfrak{a}}^+} = \mathcal{NW}(\phi^t).$$

Theorem 6 (Quint 2005)**.** *Let $\Gamma \subset G$ be Zariski-dense.*

- *The following are equivalent:*
  - *(i)* $\Omega_{\mathring{\mathfrak{a}}^+} \cap \Omega_{-\mathring{\mathfrak{a}}^+}$ *is compact;*
  - *(ii)* Γ *acts cocompactly on the union $F \subset \mathbb{X}$ of a certain "natural family" of maximal flats of $\mathbb{X}$ generalizing the union of all closed geodesics in rank 1.*
- *If the above holds, then Γ is a product of convex-cocompact subgroups of the rank 1 factors in G_1 and a uniform lattice in $G_{\geq 2}$.*

$\implies$ The generalizations of Items (3), (4) of Thm. 4 to $\mathrm{rk}(G) \geq 2$ considered by Quint are uninteresting.

There are potentially many other ways to generalize (3) and (4)! Still, Quint's observation lead people to focus on the other Items in Thm. 4.

2.3. **First definitions of Anosov representations.**

2.3.1. *Fundamental groups of neg. curved closed Riem. manifolds.* Let

$$\Gamma = \pi_1(M),$$

M a compact negatively curved Riemannian manifold.

$$\implies \Gamma \text{ hyperbolic.}$$

Γ acts on the unit tangent bundle

$$\widetilde{\mathcal{M}} = S^1\widetilde{M} \subset T\widetilde{M}$$

of the universal cover $\widetilde{M}$ of M, equipped with the lift $\widetilde{\varphi}^t$ of the geodesic flow φ^t on T^1M, which commutes with the Γ-action.

Let $P \subset G$ be a parabolic subgroup, $\bar{P}$ its opposite. Then

$$\mathcal{O} := \{(gP, g\bar{P}) \mid g \in G\}$$

is the unique open G-orbit in $G/P \times G/\bar{P}$. Have a splitting

$$T(G/P \times G/\bar{P})|_{\mathcal{O}} = \underbrace{T(G/P)|_{\mathcal{O}}}_{=:\mathscr{E}^+_{\mathcal{O}}} \oplus \underbrace{T(G/\bar{P})|_{\mathcal{O}}}_{=:\mathscr{E}^-_{\mathcal{O}}}$$

Let $\rho : \Gamma \to G$ be a group homomorphism.

Γ acts on $\widetilde{\mathcal{M}} \times \mathcal{O}$ by

$$\gamma \cdot (\xi, gP, g\bar{P}) := (\gamma \cdot \xi, \rho(\gamma)gP, \rho(\gamma)g\bar{P}).$$

$\mathbb{R}$ also acts on $\widetilde{\mathcal{M}} \times \mathcal{O}$ by

$$t \cdot (\xi, gP, g\bar{P}) := (\widetilde{\varphi}^t(\xi), gP, g\bar{P}).$$

The two actions commute $\implies$ the $\mathbb{R}$-action descends to a flow ψ^t on

$$\widetilde{\mathcal{M}} \times_\rho \mathcal{O} := (\widetilde{\mathcal{M}} \times \mathcal{O})/\Gamma$$

The differential of the Γ-quotient projection maps the Γ-invariant and $\mathbb{R}$-invariant subbundles $\mathscr{E}^\pm_{\mathcal{O}} \subset T\mathcal{O} \subset T(\widetilde{\mathcal{M}} \times \mathcal{O})$ to $d\psi^t$-invariant subbundles

$$\mathscr{F}^\pm \subset T(\widetilde{\mathcal{M}} \times_\rho \mathcal{O}).$$

$\widetilde{\mathcal{M}} \times_\rho \mathcal{O}$ is a smooth fiber bundle with fiber $\mathcal{O}$ over the unit tangent bundle

$$\mathcal{M} := \widetilde{\mathcal{M}}/\Gamma = T^1 M$$

via the projection

$$[\xi, gP, g\bar{P}] \longmapsto [\xi].$$

The flow ψ^t on $\widetilde{\mathcal{M}} \times_\rho \mathcal{O}$ lifts the geodesic flow φ^t on $\mathcal{M}$.

Definition 2.4 (Labourie 2006)**.** The homomorphism $\rho : \Gamma \to G$ is a *P-Anosov representation* if there is a continuous section

$$s : \mathcal{M} \to \widetilde{\mathcal{M}} \times_\rho \mathcal{O}$$

such that $s \circ \varphi^t = \psi^t \circ s$ and on the vector bundles $s^*\mathscr{F}^+$ (resp. $s^*\mathscr{F}^-$) over $\mathcal{M}$ the flow induced by φ^t is uniformly contracting (resp. expanding) with respect to some (hence any) bundle norms.

2.3.2. *General hyperbolic groups.*

Γ hyperbolic group, $(\widehat{\Gamma}, \Phi^t)$ a Gromov-geodesic flow of Γ

In the above, replace $(\widetilde{\mathcal{M}}, \widetilde{\varphi}^t)$ by $(\widehat{\Gamma}, \Phi^t)$ and $(\mathcal{M}, \varphi^t)$ by $(\Gamma\backslash\widehat{\Gamma}, \phi^t)$, where ϕ^t is the flow on $\Gamma\backslash\widehat{\Gamma}$ induced by Φ^t:

Definition 2.5 (Guichard-Wienhard 2012)**.** A homomorphism $\rho : \Gamma \to G$ is a *P-Anosov representation* if there is a continuous section

$$s : \Gamma\backslash\widehat{\Gamma} \to \widehat{\Gamma} \times_\rho \mathcal{O}$$

such that $s \circ \phi^t = \psi^t \circ s$ and on the vector bundles $s^*\mathscr{F}^+$ (resp. $s^*\mathscr{F}^-$) over $\Gamma\backslash\widehat{\Gamma}$ the flow induced by ϕ^t is uniformly contracting (resp. expanding) with respect to some (hence any) bundle norms.

$$\left\{\text{Continuous flow-equivariant sections } s : \Gamma\backslash\widehat{\Gamma} \to \widehat{\Gamma} \times_\rho \mathcal{O}\right\}$$

$$\overset{\cong}{\longleftrightarrow}$$

$$\left\{\ \Phi^t\text{-invariant } \Gamma\text{-equivariant maps } \widehat{\Gamma} \to \mathcal{O}\ \right\}$$

Using the defining properties of $\widehat{\Gamma}$ one deduces:

Lemma 2.6. *$\varrho : \Gamma \to G$ is P-Anosov iff there are continuous maps*

$$\xi : \partial_\infty\Gamma \to G/P, \qquad \bar{\xi} : \partial_\infty\Gamma \to G/\bar{P}$$

such that for all $(x, x') \in \partial_\infty\Gamma^{(2)}$ one has $(\xi(x), \bar{\xi}(x')) \in \mathcal{O}$ and the Gromov-geodesic flow Φ^t is exponentially contracting (resp. expanding) on the bundles $(\xi \circ \tau_+)^*\mathscr{F}^+$ (resp. $(\xi \circ \tau_-)^*\mathscr{F}^-$) over $\widehat{\Gamma}$.

$\implies$ Anosov representations generalize Item (8) of Theorem 4.

2.4. Properties of Anosov representations.

- $\mathrm{rk}(G) = 1$: ϱ $P_{\min}$-Anosov $\iff$ $\varrho(\Gamma) \subset G$ convex-cocompact.
- P-Anosov representations form an open set in the "representation variety" of all group homomorphisms $\varrho : \Gamma \to G$.
- Every P-Anosov representation $\varrho : \Gamma \to G$ is a quasi-isometric embedding, but not conversely. Item (6) of Theorem 4 does not generalize well to higher rank.
- Abundant (families of) examples.
- ϱ P-Anosov $\implies$ $\mathbb{X} := \varrho(\Gamma)\backslash G/K$ is topologically tame: There is a compactification $\overline{\mathbb{X}}$ of $\widetilde{\mathbb{X}} = G/K$ and a Γ-invariant subset $\overline{\mathbb{X}}_\varrho \subset \overline{\mathbb{X}}$ containing $\widetilde{\mathbb{X}}$ such that $\varrho(\Gamma)$ acts properly discontinuously and cocompactly on $\overline{\mathbb{X}}_\varrho$ and the compact quotient

  $$\overline{\mathbb{X}} := \Gamma\backslash\overline{\mathbb{X}}_\varrho$$

  contains $\mathbb{X}$ as a dense subset.

- Many examples of *domains of discontinuity*, i.e., open $\varrho(\Gamma)$-invariant subsets $U \subset G/H$ for appropriate H such that $\varrho(\Gamma)$ acts properly discontinuously (and possibly cocompactly) on U. However, no examples of U which are always non-empty!

- Many more equivalent characterizations of being P-Anosov.

2.5. **A minimalist definition.** Let $P_\Theta \subset G$ be a standard parabolic subgroup associated with a subset $\Theta \subset \Delta$ of a chosen simple system of restricted roots.

Definition 2.7. A finitely generated discrete subgroup $\Gamma < G$ is P_Θ-Anosov iff there are constants $c, C > 0$ such that $\alpha(\mu(\gamma)) \geq c|\gamma|_\Gamma - C$ for all $\gamma \in \Gamma$ and $\alpha \in \Theta$, where $|\cdot|_\Gamma$ denotes the word length with respect to a finite generating set.

Γ is then automatically hyperbolic.

$\implies$ Anosov representations generalize also Item (7) of Theorem 4.

2.6. **Work in progress.**

Items of Theorem 4 with colors:
Does not (seem to) generalize well to higher rank
Generalized by Anosov representations to higher rank

(1) There exists a non-empty Γ-invariant convex set $S \subset \widetilde{\mathbb{X}}$ on which Γ acts cocompactly;

(2) Γ acts cocompactly on the convex hull $\mathrm{Conv}(\Lambda_\Gamma) \subset \widetilde{\mathbb{X}}$;

(3) The closure of the union of all closed geodesics in $\mathbb{X}$ is compact;

(4) The non-wandering set $\mathcal{NW}(\phi^t)$ of the flow ϕ^t is compact;

(5) ϕ^t is an Axiom A flow;

(6) Γ is finitely generated and the inclusion $\Gamma \to G$ is a quasi-isometric embedding;

(7) Γ is finitely generated and for some word metric d_Γ on Γ there are $c, C > 0$ such that

$$d_{\widetilde{\mathbb{X}}}(\gamma K, K) \geq c\, d_\Gamma(\gamma, e) - C \qquad \forall\, \gamma \in \Gamma,$$

where $d_{\widetilde{X}}$ is the Riemannian distance in $\widetilde{\mathbb{X}}$;

(8) Γ is hyperbolic and there exists a continuous, injective, and Γ-equivariant map

$$\xi : \partial_\infty \Gamma \to G/P = \partial_\infty \mathbb{X}.$$

Q: What about the characterization of Item (5) in higher rank?

A: Work in progress, joint with Daniel Monclair and Andrew Sanders.

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